Section One: Calculator-free **35% (51 Marks)**

This section has **seven (7)** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 50 minutes.

Question 1 **(6 marks)**

(a) Determine the number of real solutions to the equation $x^2 + x + 1 = 0$. (1 mark)

$b^2 - 4ac = 1 - 4 = -3$

$\therefore$ no real sol'n

(b) Determine all complex solutions to the equation $x^2 + 2x + 10 = 0$. (2 marks)

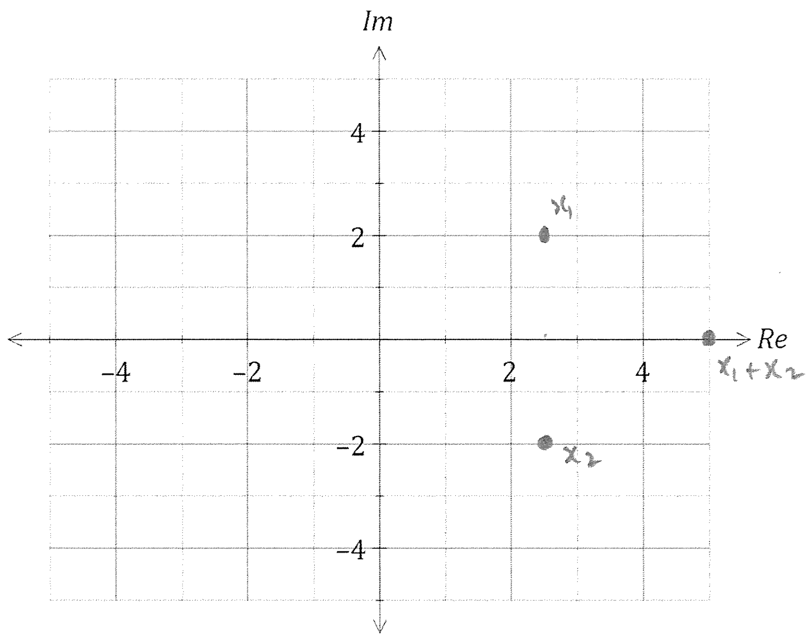
$(x+1)^2 + 9 = 0$

$(x+1)^2 = -9$

$x + 1 = \pm 3i$

$x = -1 \pm 3i$

(c) x_1 and x_2 are the complex solutions to the equation $4x^2 = 20x - 41$. If $x_1 = 2.5 + 2i$, plot x_1, x_2 and $x_1 + x_2$ in the complex plane below. (3 marks)



Question 2 **(7 marks)**

Three vectors are given by $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 3\mathbf{j}$ and $\mathbf{c} = 3\mathbf{i} + \mathbf{j}$.

$$\begin{array}{r} 16 \\ 1.\ 64 \\ \hline 80 \end{array}$$

Determine

(a) a unit vector $\mathbf{d}$, parallel to $\mathbf{a} + 2\mathbf{b}$. (3 marks)

$80 = 4 \times 20$
$= 16 \times 5$

$\underset{\sim}{a} + 2\underset{\sim}{b} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}$, $|\underset{\sim}{a} + 2\underset{\sim}{b}| = \sqrt{80} = 4\sqrt{5}$

unit vector $\hat{\underset{\sim}{d}} = \frac{1}{4\sqrt{5}} \begin{pmatrix} 4 \\ -8 \end{pmatrix}$ $\left[\frac{1}{\sqrt{80}} \begin{pmatrix} 4 \\ -8 \end{pmatrix} \right]$

$= \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \end{pmatrix}$

(b) the value(s) of k so that the magnitude of the vector $\mathbf{a} + k\mathbf{b}$ is 4. (4 marks)

$|\underset{\sim}{a} + k\underset{\sim}{b}| = \left| \begin{pmatrix} 2+k \\ -2-3k \end{pmatrix} \right|$

$= \sqrt{(2+k)^2 + (-2-3k)^2} = 4$

ie $4 + 4k + k^2 + 4 + 12k + 9k^2 = 16$

ie $10k^2 + 16k - 8 = 0$

ie $5k^2 + 8k - 4 = 0$

$(5k - 2)(k + 2) = 0$

$\therefore\ k = \frac{2}{5}$ or -2

2x1 x1x2 2x2

Question 3 **(9 marks)**

Consider the matrices $A = \begin{bmatrix} 2 & -3 \\ -2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & -5 \end{bmatrix}$.

2x2 2x1 2x3 1x2 2x3

(a) It is possible to form the product of all four matrices. State the dimensions of the resulting product. (2 marks)

ABDC = 2x3

or BDAC = 2x3

(b) Determine the matrix $\frac{1}{2}DC$. (2 marks)

$DC = \begin{bmatrix} 4 & -5 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 4 & -10 & 6 \end{bmatrix}$

$\therefore \frac{1}{2}DC = \begin{bmatrix} 2 & -5 & 3 \end{bmatrix}$

(c) Determine the inverse of matrix A. (2 marks)

$\det A = 8 - 6 = 2$

$\therefore A^{-1} = \frac{1}{2}\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1.5 \\ 1 & 1 \end{bmatrix}$

(d) Clearly show use of matrix algebra to solve the system of equations $2x - 3y + 3 = 0$ and $4y = 2x + 2$. (3 marks)

$2x - 3y = -3$
$-2x + 4y = +2$ ✓

$\Rightarrow \begin{bmatrix} 2 & -3 \\ -2 & +4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ +2 \end{bmatrix}$ ✓

$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -3 \\ 2 \end{bmatrix}$ ✓

$= \frac{1}{2}\begin{bmatrix} -6 \\ -2 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ -1 \end{bmatrix}$ ✓

Question 4 11 ~~(7 marks)~~

Let $z_1 = 2 - 2i$ and $z_2 = 3 + i$.

(a) Simplify 2

(i) $2z_1 - z_2$. (1 mark)

$= 1 - 5i$

3

(ii) z_1^3. $(z_1)^3$ (2 marks)

OR
$(2-2i)^3$
$= 2^3 + 3(2)^2(-2i) + 3(2)(-2i)^2 + (-2i)^3$
$= -16 - 16i$

$z_1^3 = 2^3(1-i)^3$

$= 8\left[(1-i)(1-i)(1-i)\right]$

$= 8\left[-2i(1-i)\right] = 8\left[-2i-2\right] = -16-16i$

(iii) $\frac{z_1}{z_2}$. (2 marks)

3

$= \frac{2-2i}{3+i} \cdot \frac{3-i}{3-i}$

$= \frac{4+i(-6-2)}{10} = \frac{4}{10} - \frac{8}{10}i$

$= \frac{2}{5} - \frac{4}{5}i$

3

(b) Show that $\overline{z_1} \times \overline{z_2} = \overline{z_1 \times z_2}$. (2 marks)

$\bar{z}_1 \times \bar{z}_2 = (2+2i)(3-i) = 8+4i$

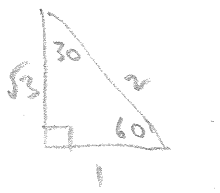
$z_1 \times z_2 = (2-2i)(3+i)$

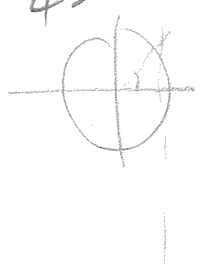
$= 8 - 4i$

$\overline{z_1 \times z_2} = 8 + 4i$

Question 5 **(~~7~~ 8 marks)**

(a) Solve the equation $\tan\left(\frac{x+25°}{2}\right) = \sqrt{3}$ for $0° \le x \le 540°$. 4 (~~3~~ marks)





$$\frac{x+25}{2} = 60°, 240°, \cancel{420}$$

$$x+25 = 120, 480°$$

$$x = \underline{95° \text{ or } 455°}$$

(b) Prove that $(1-\cos x)(1+\sec x) = \sin x \tan x$. (4 marks)

$$LHS = 1 + \sec x - \cos x - \cos x \sec x$$

$$= \frac{\cancel{\cos x} + 1 - \cos^2 x - \cancel{\cos x}}{\cos x}$$

$$= \sin^2 x \cdot \frac{1}{\cos x}$$

$$= \underline{\sin x \cdot \tan x} = RHS$$

Question 6 5 **(~~7~~ marks)**

(a) Determine the value(s) of a for which the matrix $\begin{bmatrix} a & a \\ 3 & 2a \end{bmatrix}$ is singular. (~~2~~ marks) 3

$2a(a) - 3a = 0$

$2a^2 - 3a = 0$

$a(2a-3) = 0$ $\therefore a = 0$ or $a = \frac{3}{2}$

(b) The non-singular matrix B is such that $[-3 \quad 2] \times B = [8 \quad 3]$ ① and $[2 \quad 6] \times B = [10 \quad 4]$ ②.

(i) Use these results to show that $[-1 \quad 8] \times B = [18 \quad 7]$. (2 marks)

$[-3 \ 2]B + [2 \ 6]B = [18 \ 7]$

$\{[-3 \ 2] + [2 \ 6]\}B = [18 \ 7]$

$[-1 \ 8]B = [18 \ 7]$

Question 7 **(8 marks)**

(a) Prove that the sum of any three consecutive terms of an arithmetic sequence with first term a and common difference d is always a multiple of three, for $a, d \in \mathbb{N}$. (~~3~~ marks) 2

$a, a+d, a+2d$

Sum $= 3a + 3d = 3(a+d) \Rightarrow$ multiple of 3

(b) Use mathematical induction to prove that $7^{2n-1} + 5$ is always divisible by 12, for $n \in \mathbb{N}$. (~~5~~ marks) 6

$n = 1, \quad 7^1 + 5 = 12 \quad \div 12 \quad \therefore$ true for $n = 1$ ✓

assume true for $n = k$

ie $7^{2k-1} + 5 = 12p, \quad p \in \mathbb{Z}$ ✓

$n = k+1$

ie $7^{2(k+1)-1} + 5$

$= 7^{2k-1+2} + 5$

$= 49 . 7^{2k-1} + 5$

$= (48+1)7^{2k-1} + 5$

$= 48 . 7^{2k-1} + 12p$

$= 12\left[4 . 7^{2k-1} + p\right]$ $\therefore$ true for $n = k+1$

$\therefore$ by Pr of MI

$7^{2n-1} + 5$ is divisible by 12 for $n \geq 1$

End of questions